ASYMPTOTICS OF AXIALLY SYMMETRIC FLUID FLOWS WITH A FREE BOUNDARY IN THE CASE OF DWINDLING VISCOSITY

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For high Reynolds numbers asymptotic expansions are constructed of the solution of the axially symmetric wave problem on the surface of a viscous incompressible fluid of infinite depth under the assumption that the tangential stresses on the free surface are of the order 0(1/Re). The principal terms of the asymptotic expansion are solutions of linear partial differential equations. The obtained result is then adapted to the case in which the fluid fills a bounded region whose boundary is a free surface. Some examples are given.

1. Formulation of the Problem

For the Navier–Stokes equations in the case of dwindling viscosity the nonlinear axially symmetric problem is considered on wave motion of a viscous incompressible fluid of infinite depth with applied stresses and the initial velocities field given, as well as the initial rise of the free surface:

$$\begin{gathered}\partial\mathbf{v}/\partial t+(\mathbf{v},\nabla)\mathbf{v}=-\nabla p+\varepsilon^2\Delta\mathbf{v}+\mathbf{g};\ \operatorname{div}\mathbf{v}=0;\\ \mathbf{v}=\mathbf{a};\ \zeta=\zeta_*(t=0);\ \mathbf{v}=\nabla\mathbf{v}=0(z=-\infty).\end{gathered}\tag{1.1}$$

The dynamic and kinematic conditions on the free surface $\Gamma_t: z = \zeta(r, t)$ are given by the following relations:

$$p-2\varepsilon^2\left[n_r^2\frac{\partial v_r}{\partial r}+n_z^2\frac{\partial v_z}{\partial z}+n_rn_z\left(\frac{\partial v_r}{\partial z}+\frac{\partial v_z}{\partial r}\right)\right]=p_*;\ \ (n_r^2-n_z^2)\left(\frac{\partial v_r}{\partial z}+\frac{\partial v_z}{\partial r}\right)+2n_rn_z\left(\frac{\partial v_z}{\partial z}-\frac{\partial v_r}{\partial r}\right)=T_1;\tag{1.2}$$

$$n_r\left(\frac{\partial v_\theta}{\partial r}-\frac{v_\theta}{r}\right)+n_z\frac{\partial v_\theta}{\partial z}=T_2;$$

$$\frac{\partial F}{\partial t}+v_r\frac{\partial F}{\partial r}+v_z\frac{\partial F}{\partial z}=0.$$

The dimensionless quantities appearing in (1.1) and (1.2) are related to the dimensional ones (the latter being distinguished by a prime) by the following formulas:

$$(r',z',\zeta',\zeta_*')=l\,(r,z,\zeta,\zeta_*);\quad t'=\gamma t;$$

$$(\mathbf{v}',\mathbf{a}')=\frac{l}{\gamma}(\mathbf{v},\mathbf{a});\quad (p',p_*',T_1',T_2')=\rho_0l^2\gamma^{-2}(p,p_*,T_1,T_2);$$

$$\varepsilon^2=\frac{l^2}{\nu\gamma}=1/\mathrm{Re}.$$

In the above r', z', θ' are cylindrical coordinates; $\mathbf{v}' = (v_r', v_z', v_\theta')$ is the velocity vector; p' is the hydrodynamic pressure; $\zeta'(r, z, t)$ is the rise of the free surface at the instant t; F(r, z, t) = 0 is the equation of the free surface Γ_t in an implicit form; $\mathbf{n} = (n_r, n_z, 0)$ is the normal unit vector to Γ_t; g is the gravitational acceleration; ρ_0 is the fluid density; l and γ are units of length and of time, respectively; ν is the

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kinematic viscosity coefficient; Re is the Reynolds number. It is assumed that the tangential stresses $T_1(r, t)$ and $T_2(r, t)$ on the free boundary are quantities of the order $0(\varepsilon^2)$. In view of the axial symmetry no functions depend on the angle θ.

The problem (1.1), (1.2) in its linearized formulation was analyzed in [1-5]. In the present article the asymptotic behavior of its solution in the case of (1.1), (1.2) is considered for $\varepsilon \to 0$. To solve the problem the method employed in [6] is used.

2. Construction of Asymptotic Expansion

Asymptotic expansions of the solution of the problem (1.1), (1.2) are now obtained in the form

$$\mathbf{v} \sim \sum_{k=0}^{N} \varepsilon^k \mathbf{v}_k(r, z, t) + \sum_{k=0}^{N} \varepsilon^k \mathbf{h}_k\left(\frac{\rho}{\varepsilon}, \varphi, t\right); \tag{2.1}$$

$$p \sim \sum_{k=0}^{N} \varepsilon^k p_k(r, z, t) + \sum_{k=0}^{N} \varepsilon^k q_k\left(\frac{\rho}{\varepsilon}, \varphi, t\right);$$

$$\zeta \sim \sum_{k=0}^{N} \varepsilon^k \zeta_k(\varphi, t).$$

The functions $\mathbf{v}_0$, p_0, ζ_0 are found as solutions of the wave problem on the surface of an ideal incompressible fluid of infinite depth,

$$\frac{\partial \mathbf{v}_0}{\partial t} + (\mathbf{v}_0, \nabla)\mathbf{v}_0 = -\nabla p_0 + \mathbf{g}; \quad \operatorname{div} \mathbf{v}_0 = 0; \quad \mathbf{v}_0 = \mathbf{a}; \tag{2.2}$$

$$\zeta_0 = \zeta_*(t = 0); \quad \mathbf{v}_0 = \nabla \mathbf{v}_0 = 0\,(z = -\infty); \quad p_0 = p_*;$$

$$\frac{\partial \zeta_0}{\partial t} + v_{r0}\frac{\partial \zeta_0}{\partial r} = v_{z0}\left(r, z, t \in \Gamma_t^0 : z = \zeta(r, t)\right).$$

The functions $\mathbf{v}_k$, p_k are found at the end of the first iteration [7]. By denoting the left-hand side of the system (1.1) by $P(\mathbf{V})$, where $\mathbf{V} = (v_r, v_z, v_\theta, p)$ it is required that the following relation be valid:

$$P(\mathbf{V}_N) = 0(\varepsilon^{N+1}); \tag{2.3}$$

$$\mathbf{V}_N \equiv \left(\sum_{k=0}^{N} \varepsilon^k \mathbf{v}_k, \sum_{k=0}^{N} \varepsilon^k p_k\right).$$

By setting the coefficients of $\varepsilon^0, \varepsilon^1, \ldots, \varepsilon^N$ in (2.3) equal to zero, to determine $\mathbf{v}_k$, p_k one finds linear systems of partial differential equations,

$$\frac{\partial \mathbf{v}_k}{\partial t} + \sum_{i+j=k} (\mathbf{v}_i, \nabla)\mathbf{v}_j = -\nabla p_k + \Delta \mathbf{v}_{k-2}; \tag{2.4}$$

$$\operatorname{div} \mathbf{v}_k = 0;$$

$$\mathbf{v}_k|_{t=0} = 0; \quad \mathbf{v}_k = \nabla \mathbf{v}_k = 0\,(z = -\infty);$$

$$(\mathbf{v}_{-1} \equiv 0, \quad k = 1, 2, \ldots, N).$$

The functions $\mathbf{h}_k$, $\mathbf{q}_k$ are concentrated in the neighborhood of the free boundary Γ_t, and compensate the deficiencies in the dynamic conditions (1.2) for tangential stresses. The free boundary Γ_t^0 for the ideal fluid for $t > 0$ is a surface consisting of those fluid particles which were found on it at $t = 0$. To construct the functions $\mathbf{h}_k$, q_k the traveling local coordinates (ρ, φ) are introduced [6]. Let $r = R(\varphi, t)$, $z = Z(\varphi, t)$ be parametric equations of the contour Γ_t^0 in the meridional section $\rho = \rho(r, z, t)$ this being the distance of the point (r, z) to Γ_t; $\varphi = \varphi(r, z, t)$ is the value of the parameter which corresponds to a point on Γ_t^0 nearest to (r, z); then the vector $\mathbf{X} = (r, z)$ is related to the vector $\mathbf{Y} = (R, Z)$ by the formula

$$\mathbf{X} = \mathbf{Y} + \rho \mathbf{n}_0. \tag{2.5}$$

In the above the distance ρ is measured along the inner normal to Γ_t, and $\mathbf{n} = (a_0, b_0)$ is the unit vector of the normal to Γ_t^0. It can be shown [8] that in a neighborhood of the boundary Γ_t^0 the formulas

$$\frac{\partial\varphi}{\partial r} = \delta^{-2}(1-\rho\varkappa)^{-1}\frac{\partial R}{\partial\varphi};\quad \frac{\partial\varphi}{\partial z} = \delta^{-2}(1-\rho\varkappa)^{-1}\frac{\partial Z}{\partial\varphi}; \tag{2.6}$$

$$\frac{\partial\rho}{\partial r} = -a_0 = -\delta^{-1}\frac{\partial Z}{\partial\varphi};\quad \frac{\partial\rho}{\partial z} = -b_0 = \delta^{-1}\frac{\partial R}{\partial\varphi};$$

$$\delta^2 = \left(\frac{\partial R}{\partial\varphi}\right)^2 + \left(\frac{\partial Z}{\partial\varphi}\right)^2;\quad \varkappa = \delta^{-3}\left(\frac{\partial R}{\partial\varphi}\frac{\partial^2 Z}{\partial\varphi^2} - \frac{\partial Z}{\partial\varphi}\frac{\partial^2 R}{\partial\varphi^2}\right).$$

are valid. In the above $\varkappa$ denotes the curvature of the contour Γ_t^0.

Equations are now determined which must be satisfied by the functions $\mathbf{h}_k$, q_k. Let $h_{\rho k}$, $h_{\varphi k}$, $h_{\theta k}$, $v_{\rho k}$, $v_{\varphi k}$, $v_{\theta k}$ be the components of the vectors $\mathbf{h}_k$, $\mathbf{v}_k$, respectively, in the coordinate system ρ, φ, θ. Equation (1.1) is now rewritten using local coordinates and bearing in mind that the Lamé coefficients are $H_\rho = 1$, $H_\varphi = \delta(1-\rho\varkappa)$, $H_0 = R + a_0\rho$. Then (2.1) are substituted in the obtained equations using at the same time (2.2) and (2.4). We expand the known coefficients into Taylor series in powers of ρ using for $\rho = 0$ the valid relation $\partial\rho/\partial t + \mathbf{v}_0\nabla\rho = 0$ and setting $\rho = \varepsilon s$. Setting the coefficients of $\varepsilon^0, \varepsilon^1, \ldots, \varepsilon^N$ for $\mathbf{h}_0$ successively equal to zero one obtains a system of nonlinear homogenous equations,

$$\frac{\partial h_{\varphi 0}}{\partial t} + (h_{\rho 1} + v_{\rho 1} + sa(t,\varphi))\frac{\partial h_{\varphi 0}}{\partial s} + (\delta^{-1}h_{\varphi 0} + b(t,\varphi))\frac{\partial h_{\varphi 0}}{\partial\varphi} + c_1 h_{\varphi 0} + \delta^{-1}R^{-1}\frac{\partial R}{\partial\varphi}h_{\theta 0}^2 = \frac{\partial^2 h_{\varphi 0}}{\partial s^2};$$

$$\frac{\partial h_{\theta 0}}{\partial t} + (h_{\rho 1} + v_{\rho 1} + sa(t,\varphi))\frac{\partial h_{\theta 0}}{\partial s} + (\delta^{-1}h_{\varphi 0} + b(t,\varphi))\frac{\partial h_{\theta 0}}{\partial\varphi} + c_4 h_{\theta 0} + c_3 h_{\varphi 0} + \delta^{-1}R^{-1}\frac{\partial R}{\partial\varphi}h_{\theta 0}h_{\varphi 0} = \frac{\partial^2 h_{\theta 0}}{\partial s^2};$$

$$\delta R\frac{\partial h_{\rho 1}}{\partial s} - \frac{\partial}{\partial\varphi}(Rh_{\varphi 0}) = 0;$$

$$\mathbf{h}_0|_{t=0} = 0,\quad \mathbf{h}_0|_{s=\infty} = 0;\quad \partial\mathbf{h}_0/\partial s|_{s=0} = 0.$$

Hence it follows that $h_{\theta 0} = h_{\varphi 0} = 0$. One finds from the continuity equation that $h_{\rho 0} = h_{\rho 1} = 0$. The coefficients $a(t,\varphi)$, $b(t,\varphi)$, $c_1(t,\varphi), \ldots, c_4$ are given by

$$a(t,\varphi) = \frac{\partial}{\partial\rho}\left[\frac{\partial\rho}{\partial t} + \mathbf{v}_0\nabla\rho\right]_{\rho=0};\quad b(t,\varphi) = \left[\frac{\partial\varphi}{\partial t} + \mathbf{v}_0\nabla\varphi\right]_{\rho=0};\quad c_1(t,\varphi) = \left[\delta^{-1}\frac{\partial v_{\varphi 0}}{\partial\varphi} - \varkappa v_{\rho 0}\right]_{\rho=0};$$

$$c_2(t,\varphi) = 2\delta^{-1}R^{-1}\frac{\partial R}{\partial\varphi}v_{\varphi 0}\Big|_{\rho=0};$$

$$c_3(t,\varphi) = \delta^{-1}R^{-1}\frac{\partial R}{\partial\varphi}v_{\varphi 0}\Big|_{\rho=0};$$

$$c_4(t,\varphi) = \left[\delta^{-1}R^{-1}\frac{\partial R}{\partial\varphi}v_{\varphi 0} + a_0R^{-1}v_{\rho 0}\right]_{\rho=0}.$$

Similarly for $\mathbf{h}_k$, q_k one obtains systems of linear equations of the form

$$\frac{\partial h_{\varphi k}}{\partial t} + sa\frac{\partial h_{\varphi k}}{\partial s} + b\frac{\partial h_{\varphi k}}{\partial\varphi} + c_1 h_{\varphi k} + c_2 h_{\theta k} - \frac{\partial^2 h_{\varphi k}}{\partial s^2} = F_{k-1};\quad \frac{\partial h_{\theta k}}{\partial t} + sa\frac{\partial h_{\theta k}}{\partial s} + b\frac{\partial h_{\theta k}}{\partial\varphi} + c_3 h_{\varphi k} + c_4 h_{\theta k} - \frac{\partial^2 h_{\theta k}}{\partial s^2} = N_{k-1}; \tag{2.7}$$

$$\frac{\partial q_{k+1}}{\partial s} + \left(\delta^{-1}\frac{\partial v_{\rho 0}}{\partial\rho} + 2\varkappa v_{\varphi 0}\right)_{\rho=0}h_{\varphi k} - 2a_0R^{-1}v_{\theta 0}h_{\theta k} = M_{k-1};$$

$$\delta R\frac{\partial h_{\rho,k+1}}{\partial s} - \delta(a_0 - \varkappa R)\frac{\partial}{\partial s}(sh_{\rho k}) - \delta a_0\varkappa\frac{\partial}{\partial s}(s^2 h_{\rho,k-1}) - \frac{\partial}{\partial\varphi}(Rh_{\varphi k}) + s\frac{\partial}{\partial\varphi}(a_0 h_{\varphi,k-1}) = 0;$$

$$\mathbf{h}_k|_{t=0} = 0,\quad \mathbf{h}_k|_{s=\infty} = q_k|_{s=\infty} = 0,\quad (k = 1, 2, \ldots, N).$$

The right-hand sides, namely, F_{k-1}, N_{k-1}, M_{k-1} are known and are expressed in terms of $\mathbf{v}_0, \ldots, \mathbf{v}_{k-1}$, $\mathbf{h}_0, \ldots, \mathbf{h}_{k-1}$. In particular, $F_0 = M_0 = N_0 = 0$ as well as $q_1 = q_0 = 0$.

Moreover, one determines equations which are satisfied by the functions $\zeta_k(t, \varphi)$. Let $\rho = \zeta(t, \varphi, \varepsilon) - \sum_{k=0}^{N} \varepsilon^k \zeta_k(t, \varphi)$ be an equation of Γ_t where $\zeta_0 = 0$ since $\rho = 0$ is the equation of Γ_t^0. One now sets $F = -\rho + \zeta$ in (1.2) and employing the same considerations as in the derivation of (2.7), one obtains

$$\frac{\partial \zeta_k}{\partial t} + b(t, \varphi)\frac{\partial \zeta_k}{\partial \varphi} - a(t, \varphi)\zeta_k = [h_{\rho k} + v_{\rho k}]_{\rho=0} + E_{k-1};\ \zeta_k|_{t=0} = 0;\ E_0 = E_1 = 0\ (k = 1,2,\ldots,N). \tag{2.8}$$

Applying simultaneously the first and the second iteration to the dynamic conditions (1.2), one finds for the systems (2.4) and (2.7) the boundary conditions for s = 0:

$$\frac{\partial h_{\varphi k}}{\partial s} = (a_0^2 - b_0^2)\left(\frac{\partial v_{r,k-1}}{\partial z} + \frac{\partial v_{z,k-1}}{\partial r}\right) + 2a_0 b_0\left(\frac{\partial v_{t,k-1}}{\partial z} - \frac{\partial v_{r,k-1}}{\partial r}\right) + A_{k-1}; \tag{2.9}$$

$$\frac{\partial h_{\theta k}}{\partial s} = a_0\left(\frac{\partial v_{\theta,k-1}}{\partial r} - \frac{v_{\theta,k-1}}{r}\right) + b_0\frac{\partial v_{\theta,k-1}}{\partial z} +$$

$$+ B_{k-1};\ p_k + q_k = 2a_0^2\frac{\partial v_{r,k-2}}{\partial r} +$$

$$+ 2b_0^2\frac{\partial v_{z,k-2}}{\partial z} + 2a_0 b_0\left(\frac{\partial v_{r,k-2}}{\partial r} + \frac{\partial v_{z,k-2}}{\partial z}\right) + D_{k-1}.$$

In the above $A_0 = B_0 = D_0 = D_1 = 0$, A_{k-1}, B_{k-1}, D_{k-1} are known being expressed in terms of $\mathbf{v}_0, \ldots, \mathbf{v}_{k-1}$, $\mathbf{h}_0, \ldots, \mathbf{h}_{k-1}$. It is noted that in this case one has $\mathbf{v}_1 = p_1 = \zeta_1 = 0$.

3. Solution of Boundary-Layer Equations

It is assumed that the solution to the problem (2.2) is known. To obtain an explicit expression for the principal terms of the asymptotic expansion for $h_{\varphi 1}$, $h_{\theta 1}$ one changes the variables in (2.7) for k = 1 by means of

$$\xi = sL(t, \varphi);\ \eta = \eta(t, \varphi);\ t_1 = t,$$

where $L(t, \varphi)$, $\eta(t, \varphi)$ are solutions of the following Cauchy problem:

$$\frac{\partial L}{\partial t} + b(t, \varphi)\frac{\partial L}{\partial \varphi} - a(t, \varphi)L = 0,\ L|_{t=0} = 1;$$

$$\frac{\partial \eta}{\partial t} + b(t, \varphi)\frac{\partial \eta}{\partial \varphi} = 0,\ \eta|_{t=0} = \varphi.$$

The first two equations of (2.7) now become

$$\frac{\partial h_{\varphi 1}}{\partial t} + c_1 h_{\varphi 1} + c_2 h_{\theta 1} = L^2\frac{\partial^2 h_{\varphi 1}}{\partial \xi^2};$$

$$\frac{\partial h_{\theta 1}}{\partial t} + c_3 h_{\varphi 1} + c_4 h_{\theta 1} = L^2\frac{\partial^2 h_{\theta 1}}{\partial \xi^2};$$

$$\mathbf{h}_1|_{t=0} = \mathbf{h}_1|_{\xi=\infty} = 0;$$

$$\left.\frac{\partial h_{\varphi 1}}{\partial \xi}\right|_{\xi=0} = \omega_1(t, \eta);\ \left.\frac{\partial h_{\theta 1}}{\partial \xi}\right|_{\xi=0}\omega_2(t, \eta),$$

where the notation

$$\omega_1(t, \eta) = L^{-1}\left[(a_0^2 - b_0^2)\left(\frac{\partial v_{r0}}{\partial z} + \frac{\partial v_{z0}}{\partial r}\right) + 2a_0 b_0\left(\frac{\partial v_{z0}}{\partial z} - \frac{\partial v_{r0}}{\partial r}\right)\right]_{\rho=0};$$

$$\omega_2(t, \eta) = L^{-1}\left[a_0\left(\frac{\partial v_{\theta 0}}{\partial r} - \frac{v_{\theta 0}}{r}\right) + b_0\frac{\partial v_{\theta 0}}{\partial z}\right]_{\rho=0}.$$

is used. Other functions $H_1(\xi, \eta, t)$, $H_2(\xi, \eta, t)$ are now introduced by means of the formulas $h_{\varphi 1} = f_1(t, \eta)H_1$, $h_{\theta 1} = f_2(t, \eta)H_2$ where f_1 and f_2 are the solutions of the system of equations

$$\partial f_1/\partial t + c_1 f_1 + c_3 f_1^2 f_2^{-1} = 0, \quad f_1|_{t=0} = 1;$$
$$\partial f_2/\partial t + c_4 f_2 + c_2 f_2^2 f_1^{-1} = 0, \quad f_2|_{t=0} = 1. \tag{3.1}$$

By introducing another variable $t_2(dt_2 = L^2 dt_1)$ one obtains for the functions $w_1(\xi, \eta, t) = H_1 + H_2$ the best-conduction equation,

$$\frac{\partial w_1}{\partial t_2} = \frac{\partial^2 w_1}{\partial \xi^2}; \quad w_1|_{t=0} = w_1|_{\xi=\infty} = 0; \quad \left.\frac{\partial w_1}{\partial \xi}\right|_{\xi=0} = \frac{\omega_1}{f_1} + \frac{\omega_2}{f_2}. \tag{3.2}$$

The solution of Eq. (3.2) can be given in a closed form, namely,

$$w_1(\xi, \eta, t) = -\int_0^{t_2} [\pi(t_2 - u)]^{-1/2} e^{-\frac{\xi^2}{4(t_2-u)}} \left[\frac{\omega_1(u, \eta)}{f_1(u, \eta)} + \frac{\omega_2(u, \eta)}{f_2(u, \eta)}\right] du. \tag{3.3}$$

The functions $H_3(\xi, \eta, t)$ and $H_4(\xi, \eta, t)$ are introduced by means of the formulas $h_{\varphi 1} = f_3(t, \eta) H_3$, $h_{\theta 1} = f_4(t, \eta) H_4$ where f_3 and f_4 are solutions of the system (3.1) in which c_1, c_3 should be replaced by c_1 and c_3, respectively. Then the function $w_2(\xi, \eta, t) = H_3 + H_4$ satisfies (3.2) provided f_1 is replaced by f_3 and f_2 by f_4, and is of the form

$$w_2(\xi, \eta, t) = \int_0^{t_2} [\pi(t_2 - u)]^{-1/2} e^{-\frac{\xi^2}{4(t_2-u)}} \left[\frac{\omega_2(u, \eta)}{f_4(u, \eta)} - \frac{\omega_1(u, \eta)}{f_3(u, \eta)}\right] du. \tag{3.4}$$

To obtain an expression for $h_{\varphi 1}$ one has to multiply (3.3) by f_4 and (3.4) by f_2 and add up. An expression for $h_{\theta 1}$ is obtained similarly.

4. The Case of Bounded Region. Examples

The method of asymptotic series described in Sec. 2 can also be adapted to the case of the fluid filling a bounded domain whose boundary is a free surface. Asymptotic expansions are now obtained in the form of (2.1). The functions v_k, p_k satisfy the systems (2.4). The formulas (2.5) and (2.6) are satisfied for the local coordinates and the functions h_k, q_k, ζ_k can be determined from (2.7)-(2.9).

Example 1. Let there be a fluid inside the ball $x^2 + y^2 + z^2 \leq 1$ at the initial time instant and let a velocity field $v_\theta = 0$; $v_r = \lambda r/\sqrt{3}$; $v_z = -2\lambda z/\sqrt{3}$ be specified therein, the ball surface being a free boundary. The corresponding flow of an ideal fluid in the absence of gravitational forces was obtained by Ovsyannikov in [9]. With t increasing the ball is deformed and becomes an ellipsoid of revolution with semiaxes given by $\tau(t)$, $\tau(t)$, $\tau^{-2}(t)$; moreover, if $\lambda < 0$, then for $t \to \infty$, $\tau \to 0$ the ellipsoid is stretched along the z-axis, and if $\lambda > 0$ the ellipsoid flattens out towards the plane $z = 0$. If the viscosity is taken into account, this results in the expansions (2.1) in which $\mathbf{v}_0$, p_0 are found from (2.2) and are given by [9]

$$v_{r0} = \dot{\tau}\tau^{-1} r; \; v_{z0} = -2\dot{\tau}\tau^{-1} z; \; v_{\theta 0} = 0; \; \dot{\tau} = d\tau(t)/dt;$$

$$p_0 = -0{,}5\tau\ddot{\tau}(r^2\tau^{-2} + z^2\tau^4 - 1);$$

$$\int_1^{\tau} \sqrt{2 + \alpha^6} \cdot \alpha^{-3} d\alpha = \lambda t \quad (\lambda = \text{const}).$$

The principal terms of the asymptotic expansions can be determined from Eqs. (2.7) and (2.9) for k = 1, where

$$Z(t, \varphi) = \tau^{-2} \sin \varphi; \; R(t, \varphi) = \tau \cos \varphi; \; a_0 = \tau^{-2}\delta^{-1} \cos \varphi;$$

$$b_0 = \tau\delta^{-2} \sin \varphi; \; \delta^2 = \tau^2 \sin^2 \varphi + \tau^{-4} \cos^2 \varphi; \; b(t, \varphi) = 0;$$

$$a(t, \varphi) = \dot{\tau}\tau^{-1}\delta^{-2}(\tau^{-4} \cos^2 \varphi - 2\tau^2 \sin^2 \varphi);$$

$$c_1 = \dot{\tau}\tau^{-1}\delta^{-2}(\tau^2 \sin^2 \varphi - 2\tau^{-4} \cos^2 \varphi);$$

$$c_3 = 0; \; \omega_1 = -3\dot{\tau}\tau^{-3}\delta^{-3} \sin 2\varphi; \; \omega_2 = 0.$$

Using the formulas (3.3) and (3.4) one finds

$$h_{\theta 1}=0,\ h_{\varphi 1} = 3\delta^{-2}\sin 2\varphi \int_0^t \frac{\dot{\tau}(u)}{\tau(u)\sqrt{\pi(t-u)}} e^{-\frac{s^2\tau^2\delta^2}{4(t-u)}} du.$$

$$h_{\theta 1}=0.$$

From (2.7) the pressure is obtained in the boundary layer, namely,

$$q_2 = 30\dot{\tau}\tau^{-3}\delta^{-4}\cos^2\varphi\sin^2\varphi \int_0^t \dot{\tau}(u)\tau^{-1}(u)\,\mathrm{erfc}\left(\frac{s\tau\delta}{2\sqrt{t-u}}\right) du.$$

The second approximation of the first iteration of $\mathbf{v}_2 = (v_{r2}, v_{z2}, 0)$ is now determined. By employing the relations $\mathbf{v}_2 = 0$, div $\mathbf{v}_2 = 0$ the function Φ is introduced. Since $\mathbf{v}_2$ = grad Φ one obtains from (2.4) a system of equations for Φ and p_2:

$$\frac{\partial\Phi}{\partial t} + \dot{\tau}\tau^{-1} r \frac{\partial\Phi}{\partial r} - 2\dot{\tau}\tau^{-1} z \frac{\partial\Phi}{\partial z} + p_2 = 0;\quad \nabla^2\Phi = 0;\quad \Phi|_{t=0} = 0;$$

$$\left(\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right);$$

$$[p_2 + q_2]_{\rho=0} = 2\dot{\tau}\tau^{-1}\delta^{-2}\left(\tau^{-4}\cos^2\varphi - 2\tau^2\sin^2\varphi\right);$$

$$\left(r, z, t \in D_t^0 : r^2\tau^{-2} + z^2\tau^4 \leqslant 1\right).$$

Hence by eliminating p_2 one obtains a boundary-value problem for Φ:

$$\nabla^2\Phi = 0;\quad \Phi|_{\rho=0} = \frac{12\tau^6\ln\tau}{(1-\tau^6)\left[1+\tau^6+(1-\tau^6)\cos 2\varphi\right]}$$

$$\left(r, z, t \in D_t^0\right).$$

The solution of the above is represented in series form,

$$\Phi = \frac{12\tau^6\ln\tau}{1-\tau^6}\sum_{k=1}^{\infty}(-1)^k G_k(\tau)\,Q_k^{-1}\left(\frac{1}{\sqrt{1-\tau^6}}\right) p_k(\cos 2\varphi)\,Q_k(\mathrm{ch}\,\alpha),$$

where $p_k(x)$, $Q_k(x)$ are the Legendre functions of the first and second kind; respectively, $G_k(\tau)$ and α being found from the relations

$$G_k(\tau) = \frac{2k+1}{2}\int_{-1}^{1}\frac{(1+\tau)\,p_k(x)}{1+\tau+x(1-\tau)}dx;\quad \frac{r^2}{\mathrm{sh}^2\alpha} + \frac{z^2}{\mathrm{ch}^2\alpha} = \frac{1-\tau^6}{\tau^4}.$$

One can now determine ζ_2 from Eq. (2.8) for k = 1, where $h_{\rho 2}$ and $v_{\rho 2}$ are bound by employing the relations

$$v_{\rho 2} = \delta^{-1}\frac{\partial\Phi}{\partial\alpha};\quad \delta R\frac{\partial h_{\rho 2}}{\partial s} + \frac{\partial}{\partial\varphi}(Rh_{\varphi 1}) = 0;\quad h_{\rho 2}|_{s=\infty} = 0.$$

Hence it follows that

$$\zeta_2 = 1.5\tau^{-2}\delta^{-1}\int_1^{\tau^2}\frac{x^3(1+\sin^2\varphi)\sin^2\varphi - (1+\cos^2\varphi)\cos^2\varphi}{(\cos^2\varphi + x^3\sin^2\varphi)^2}\sqrt{2+x^3}\ln x\,dx + 3\tau^{-1}\delta^{-1}\sum_{k=1}^{\infty}\alpha_k(\tau)\,p_k(\cos 2\varphi);$$

$$\alpha_k(\tau) = (-1)^k\int_1^{\tau^2}\frac{x^3\ln x}{1-x^6}G_k(x^3)\,Q_k^{-1}\left(\frac{1}{\sqrt{1-x^3}}\right)\frac{d}{dx}Q_k\left(\frac{1}{\sqrt{1-x^3}}\right)\sqrt{\frac{2+x^3}{1-x^3}}dx.$$

Example 2. The free ascent of a gas cavity in fluid is considered. It is assumed that the cavity radius c is so small that subjected to surface tension it retains its spherical shape. We locate the origin of the traveling coordinate system at the center of the cavity; then in this coordinate system the flow function ψ_0 of the degenerate problem (2.2) for the flow outside the cavity which moves with velocity u(t) is given by the expression

$$\psi_0 = \frac{1}{2} u(t)\left(r^2 - \frac{c^3}{r}\right)\sin^2\varphi.$$

In this example, r, φ, θ are always spherical coordinates. The flow takes place in a meridional section, therefore $v_{k\theta} = h_{k\theta} = 0$. The equation and the boundary conditions for the velocity $h_{\varphi 1}$ in the boundary layer on the cavity are given by

$$\frac{\partial h_{\varphi 1}}{\partial t} - 2\beta s\frac{\partial h_{\varphi 1}}{\partial s}\cos\varphi + \beta\frac{\partial h_{\varphi 1}}{\partial\varphi}\sin\varphi + \beta h_{\varphi 1}\cos\varphi = \frac{\partial^2 h_{\varphi 1}}{\partial s^2};$$

$$h_{\varphi 1}|_{t=0} = h_{\varphi 1}|_{s=\infty} = 0;$$

$$\left.\frac{\partial h_{\varphi 1}}{\partial s}\right|_{s=0} = 2\beta\sin\varphi \quad \left(\beta = \frac{3}{2}\frac{u(t)}{c}\right).$$

Applying the method used in Sec. 3 one obtains an expression for $h_{\varphi 1}$ and from (2.7) for k = 1 the pressure q_2 is found in the boundary layer:

$$h_{\varphi 1} = \frac{2}{\sqrt{\pi}\cdot\sin\varphi}\int_0^\tau \frac{B(u,\varphi)}{\sqrt{\tau - u}}\, e^{-\frac{s^2\sin^2\varphi}{4(\tau - u)}}\,du;$$

$$q_2 = -12uc^{-2}\sin^{-2}\varphi\int_0^\tau B(u,\varphi)\operatorname{erfc}\left(\frac{s\sin^2\varphi}{2\sqrt{\tau - u}}\right)du,$$

where

$$\tau = 16e^{-2\int_0^t \beta(t)dt}\,\mathrm{tg}^2\frac{\varphi}{2}\int_0^t \frac{\exp\left(4\int_0^x \beta(x)\,dx\right)dx}{\left[1 + \exp\left(2\int_0^x \beta(x)\,dx - 4\int_0^t \beta(t)\,dt\right)\mathrm{tg}^2\frac{\varphi}{2}\right]^4}.$$

In the above B(u, φ) is obtained from $\beta(t)$ if the last formula in which τ is replaced by u is employed.

In the case u = const the formulas are consistent with the result obtained by Petrov in [10].

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